CAP representations
(The right(?) way for generic MR analysis of Internet data)

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WISP: UCSD, November 2004

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*: *Breaking news, 06/11/04: Wisconsin routed Minnesota 38:14, on its way to the national title.*

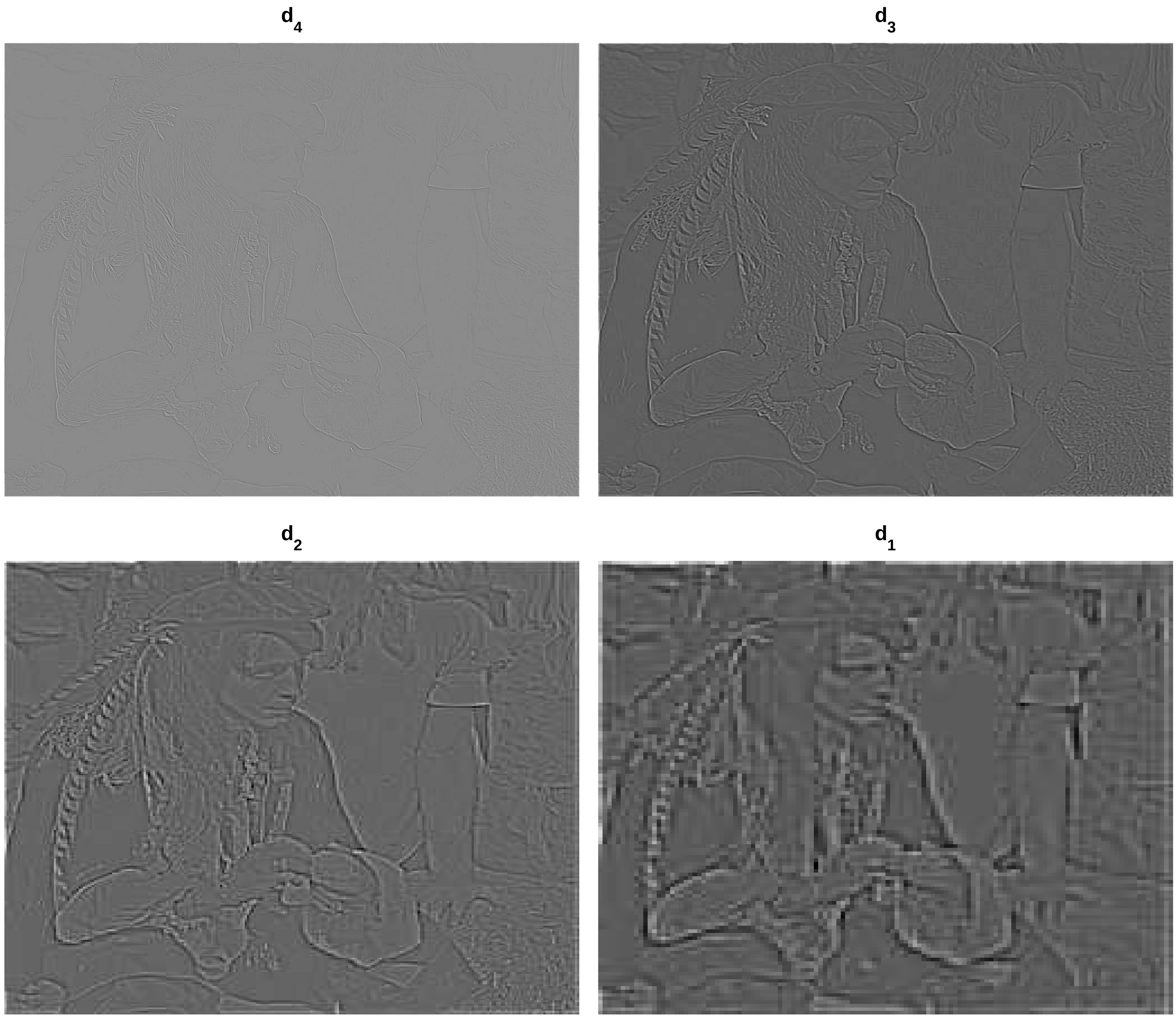
Outline

- Possible goals behind "generic analysis on Internet signals"
- Why is that a non-trivial task?
- Predictability and pyramidal algorithms
- Performance of pyramidal representation
- CAMP and my favorite pyramidal representation
- What parameters to extract from the representation?

A mathematical view of Internet signals

- Main features in the signal:
  - burst types
  - rate of their appearances
- This is non-trivial
  (why? After all, nothing is easier than 1D signals...)
  - the amount of data may be overwhelming
  - there is no clear way to judge success
- It is also a cultural problem. really?

maybe it is time to show some images?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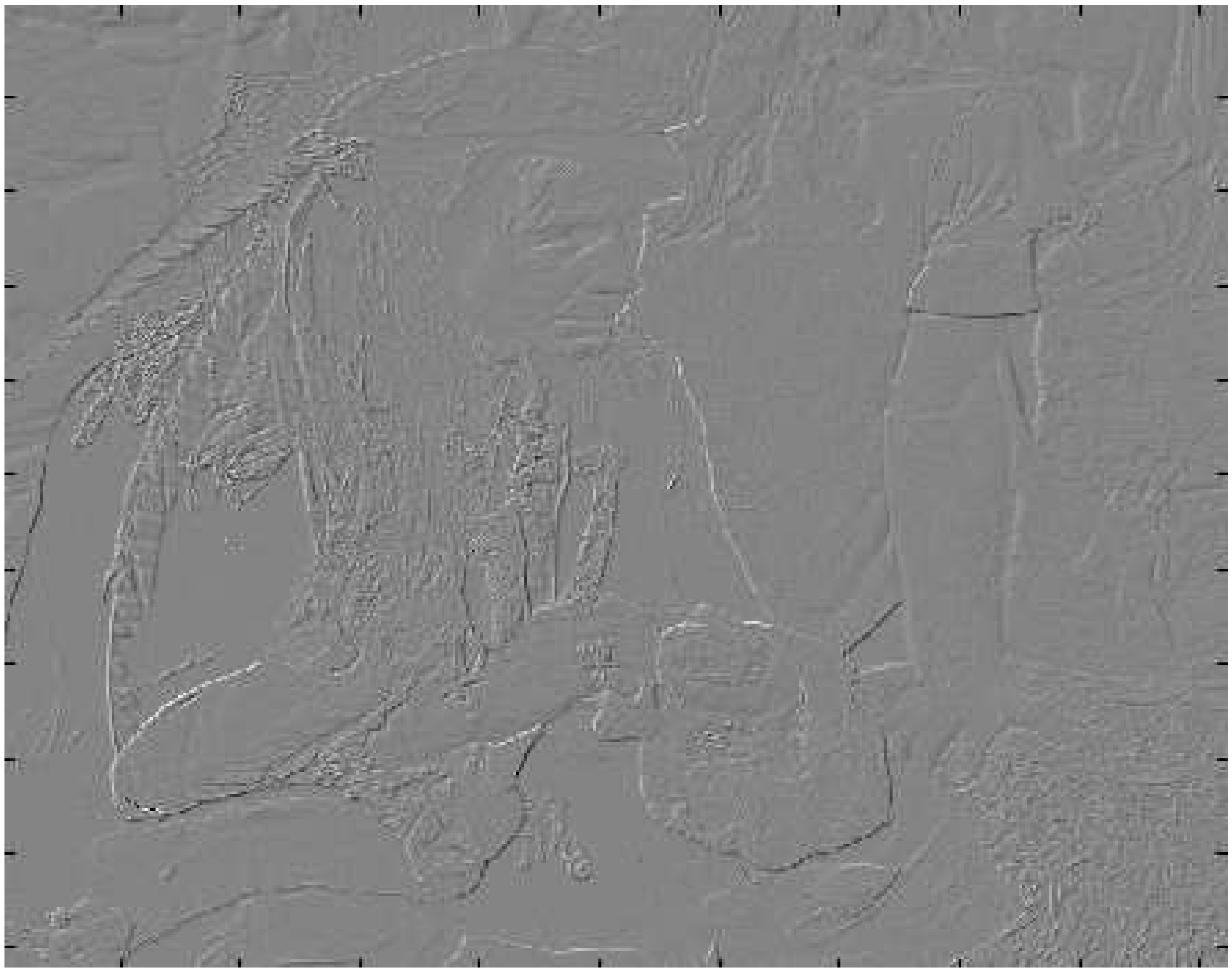

Pyramidal algorithms I: MR representation

$h : \mathbb{Z} \to \mathbb{R}$ is a symmetric, normalized, filter:
$h(k) = h(-k)$, $\sum_{k\in\mathbb{Z}} h(k) = 1$.

$\downarrow$, $\uparrow$ are downsampling & upsampling:

$$\begin{aligned} y_{\downarrow}(k) &= y(2k), \quad k \in \mathbb{Z} \\ y_{\uparrow}(k) &= \begin{cases} 2y(k/2), & k \text{ even}, \\ 0, & \text{otherwise}. \end{cases} \end{aligned}$$

$(y_j)_{j=-\infty}^{\infty} \subset \mathbb{C}^{\mathbb{Z}}$ s.t:

$$y_j = Cy_{j+1} := (h * y_j)_{\downarrow}, \quad \forall j.$$

C is Compression or Coarsification

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y_{j+1} is then **predicted** from y_j by

$$y_{j+1} \approx Py_j := h * (y_{j\uparrow}).$$

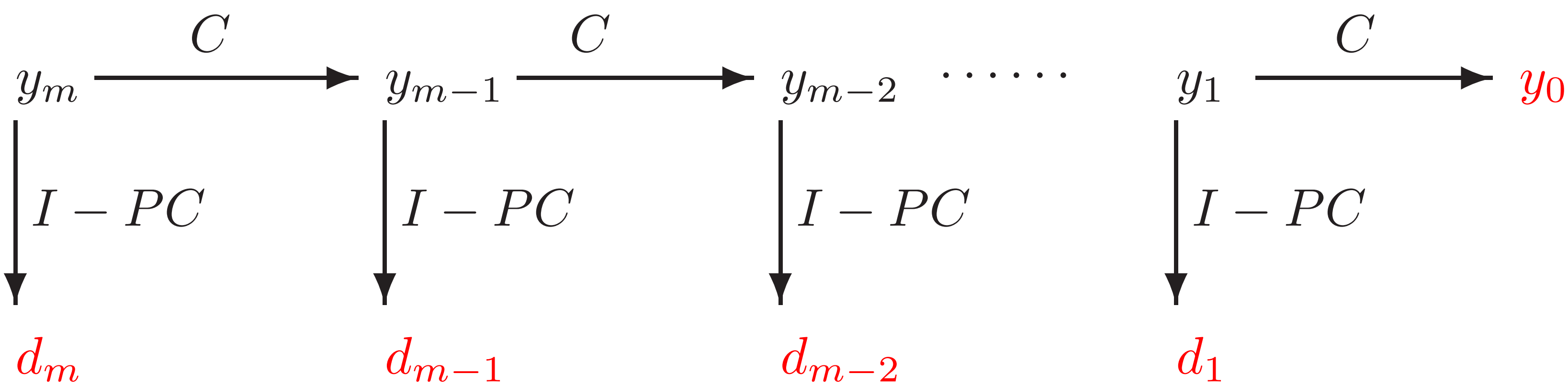
P is Prediction or subdivision

Pyramidal algorithms II: the detail coefficients

- Define the detail coefficients:

$$d_j := (I - PC)\, y_j = y_j - P\, y_{j-1}.$$

- Replace y_j by the pair $(y_{j-1},\, d_j)$.
- Continue iteratively.



Reconstruction. Recovering y_m from $y_0, d_1, d_2, \ldots, d_m$ is trivial:

$y_1 = d_1 + P y_0$, $y_2 = d_2 + P y_1$ and so on.

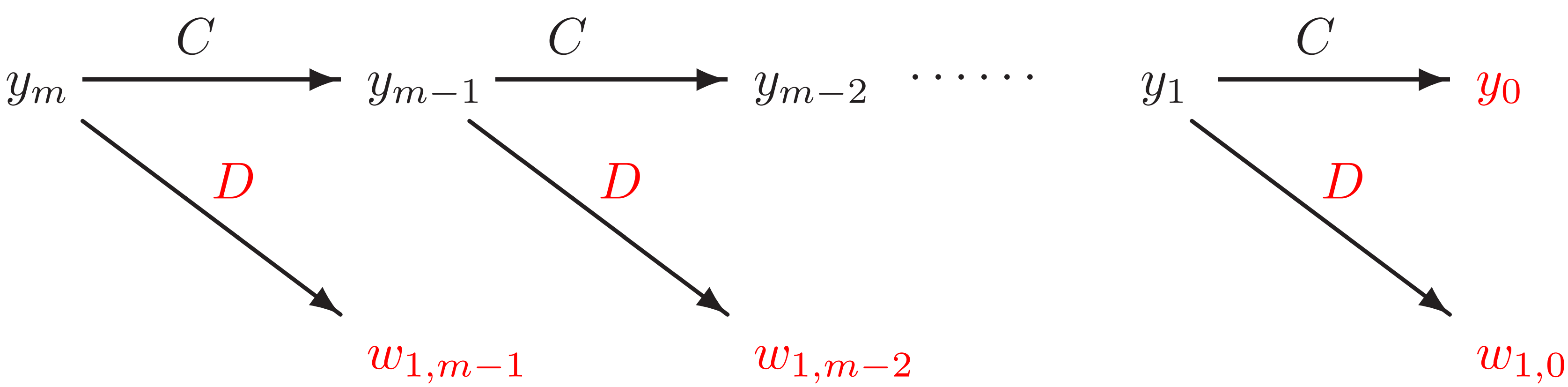
Wavelet pyramids, Mallat, 1987

Decompose the detail map $I - PC$:

$$I - PC = RD$$

$$D : y \mapsto (h_1 * y)_\downarrow =: w_{1,j-1}, \quad R : y \mapsto h_1 * y_\uparrow$$

with h_1 a real, symmetric, highpass: $\sum_{k \in \mathbb{Z}} h_1(k) = 0$.

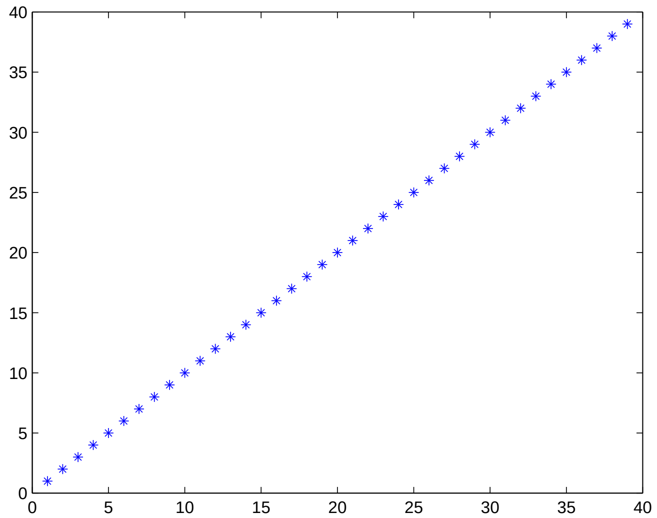


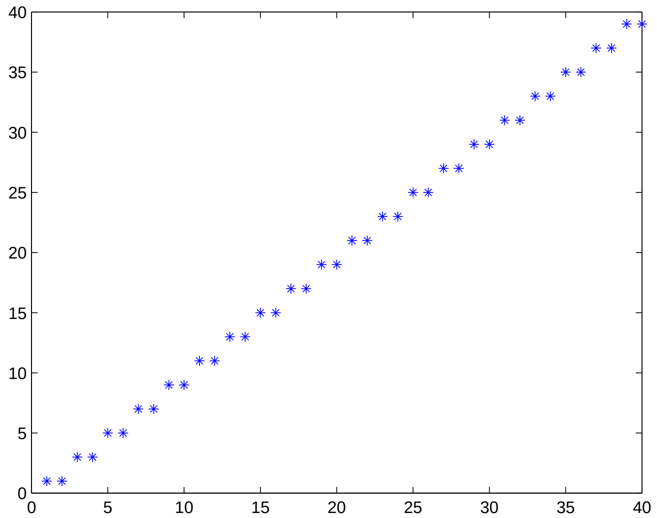
Note that we can recover y_m from $y_0, w_{1,0}, w_{1,1}, \ldots, w_{1,m-1}$ since $y_1 = Rw_{1,0} + Py_0$, $y_2 = Rw_{1,1} + Py_1$ and so on.

Performance

- Ability to predict. The best prediction are based on local averaging, and on nothing else =spline predictors
- Time blurring: good prediction requires long averaging filter. That blurs spontaneous events.
- Internet data exhibit different behaviour at "small" scales than other scales. Hence: non-stationary representation
- Standard wavelet systems are mediocre for Internet data: they blur time, and create artifacts, in order to gain unnecessary properties (orthonormality).

Poor prediction





My favorite representation

well, before we conducted any numerical tests

Step I: Build an MR representation based on

$$h_1 = \frac{1}{4}(1\ 2\ 1)$$

Step II: Define the detail coefficients by:

$$d_j(k) = \begin{cases} \frac{-y_j(k+1)+2y_j(k)-y_j(k-1)}{4}, & k \text{ even}, \\ \\ \frac{y_j(k-3)-9y_j(k-1)+16y_j(k)-9y_j(k+1)-y_j(k+3)}{16}, & k \text{ odd}. \end{cases}$$

The "performance grade" here is 2 in the strict sense. (To compare, Haar's grade is 1 in the non-strict sense, and 0 in the strict sense.)

This is an example of a new class of high-performance representations called CAMP

what to analyse? what to extract?

for $p \geq 1$, the p-norm is

$$\|a\|_p = (\sum_k |a_k|^p)^{1/p}$$

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the best thing to analyse is the "compressibility" of the detail coefficients: choose a number N, then

(1) replace the N "most important" detail coefficients by 0, to obtain a signal e_N.

(2) reconstruct using e_N to obtain Y_N. '

(3) define $e_p(N) := \|Y_N\|_p$.

(4) find the a parameter α such that

$$e_p(N) \approx CN^{-\alpha}.$$

$\alpha(p)$ = the predictability parameter in the p-norm

"most important"=?

(1) Non-linear: choose the largest ones

(2) Linear: go from coarse scale to fine scale.

Output: this way we have two functions $p \mapsto \alpha(p)$.

Goal: learn how to judge properties of your signal based on these two functions

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it might be that the detail coefficients behave rather differently at different scale (small scale vs. large scale).

CAP representations

Choose:

- two refinable functions ϕ_c, ϕ_r with refinement filters h_c, h_r.
- A third (Auxiliary-Alignment) lowpass filter h_a.

Decompose: Fix $f : \mathbb{R} \to \mathbb{C}$.
For all $k, j \in \mathbb{Z}$, define $y_j(k) := 2^{j/2} \langle f, (\phi_c)_{j,k} \rangle$.
The CAP operators are:

$$
\begin{aligned}
C : y \mapsto (h_c * y)_{\downarrow}, &\qquad \text{(Coarsification-Compression)},\\
A : y \mapsto Ay := h_a * y, &\qquad \text{(Alignment)},\\
P : y \mapsto Py := h_r * (y_{\uparrow}), &\qquad \text{(Prediction-subdivision)}.
\end{aligned}
$$

Then $Cy_{j+1} = y_j$, $\forall j$.

The detail coefficients are:

$$d_j := (A - PAC)y_j = Ay_j - PAy_{j-1}.$$

This is the **CAP representation** with (d_j) the **CAP coefficients**.

$$\begin{array}{ccccccc}
y_m & \xrightarrow{C} & y_{m-1} & \xrightarrow{C} & y_{m-2} \quad \cdots\cdots & y_1 & \xrightarrow{C} \; y_0 \\
\downarrow{\scriptstyle A-PAC} & & \downarrow{\scriptstyle A-PAC} & & \downarrow{\scriptstyle A-PAC} & \downarrow{\scriptstyle A-PAC} & \\
d_m & & d_{m-1} & & d_{m-2} & d_1 &
\end{array}$$

y_m is recovered from $y_0, d_1, d_2, \ldots, d_m$ since

$Ay_1 = d_1 + PAy_0, \quad Ay_2 = d_2 + PAy_1, \quad \ldots, \quad Ay_m = d_m + PAy_{m-1}$

and deconvolving A from Ay_m.

Summary

Do they

	W	F	CAP
implemented by fast pyramid algorithms ?	✓	✓	✓
provides good function space characterizations ?	✓	✓	✓
avoid mother wavelets ?			✓
very short filters, with no artifacts ?		✓	✓
have simple constructions ?		✓	✓
avoid redundant representations ?	✓		

Wavelet are non-redundant. Caplets are only slightly redundant in high dimensions. Their redundancy is non-essential.

CAMP representations: Compression-Alignment-Modified Prediction

With CAP in hand, one can modify the process s.t.:

- The filters are shorter
- The performance (:= function space characterization) is the same

Example: Assume h is interpolatory. Define the details as:

$$d_j := \begin{cases} y_j - h * y_j, & \text{on } 2\mathbb{Z}^d, \\ y_j - h * (y_{j\downarrow\uparrow}), & \text{otherwise.} \end{cases}$$

Let ϕ be the refinable function of h. If

$$\phi \in C_c^{s+\epsilon},$$

then the above detail characterize L_p^s.

Example (2D): $h = \begin{bmatrix} 0 & 1/8 & 1/8 \\ 1/8 & 1/4 & 1/8 \\ 1/8 & 1/8 & 0 \end{bmatrix}$.

There are four (hidden) filters, for computing d_j:

$$\begin{bmatrix} 0 & -1/8 & -1/8 \\ -1/8 & +3/4 & -1/8 \\ -1/8 & -1/8 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 \\ -1/2 & +1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1/2 & 0 \\ 0 & +1 & 0 \\ 0 & -1/2 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & -1/2 \\ 0 & +1 & 0 \\ -1/2 & 0 & 0 \end{bmatrix}$$

Those are $7, 3, 3, 3$-tap. There are four (hidden) "CAMPlets", whose average area of support is about 2.

The performance is on par with tensor $3/5$, whose filters are $25, 15, 15$-tap. Each supported in 3×3 square.

$\mathbf{y_j}$

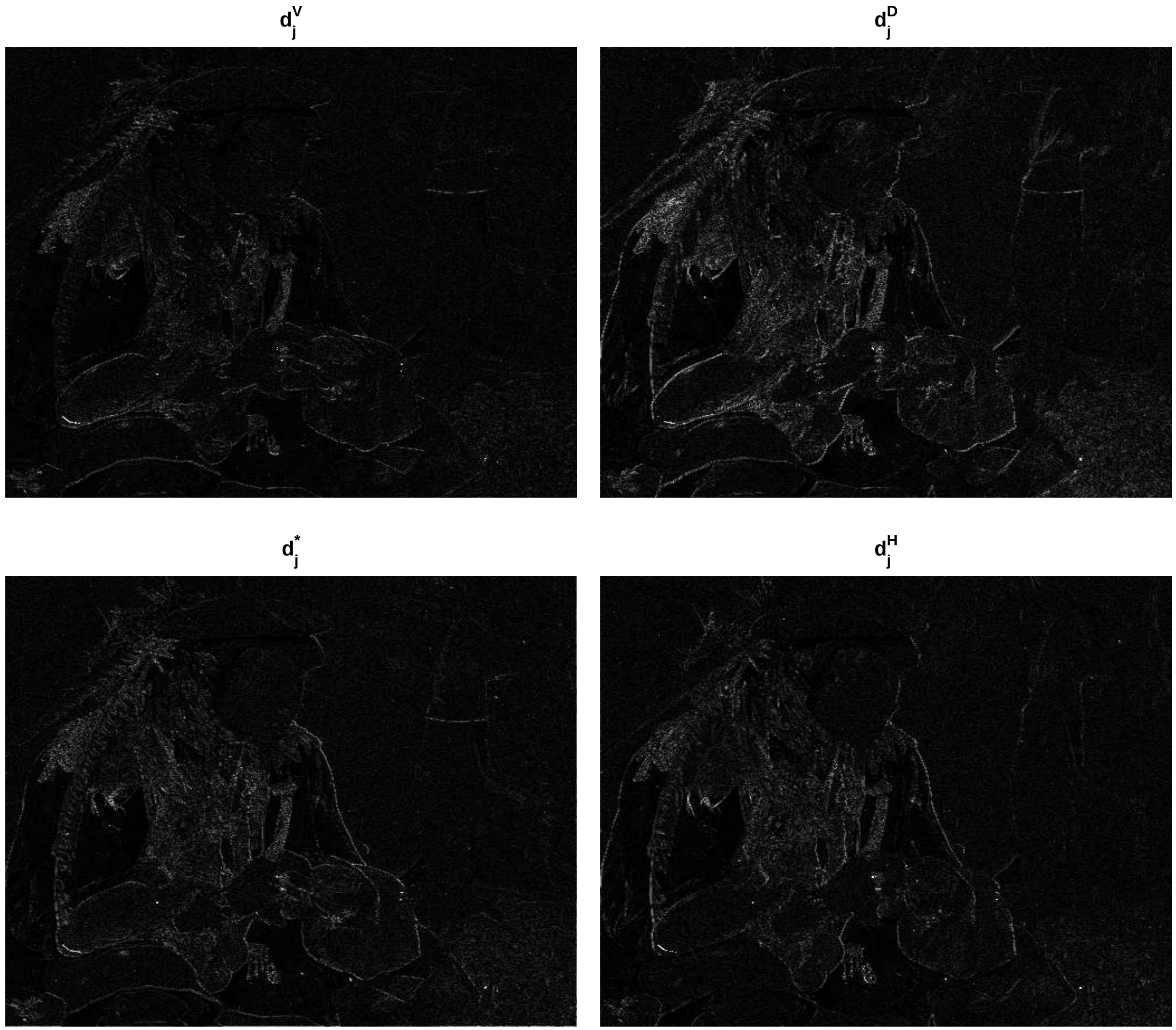
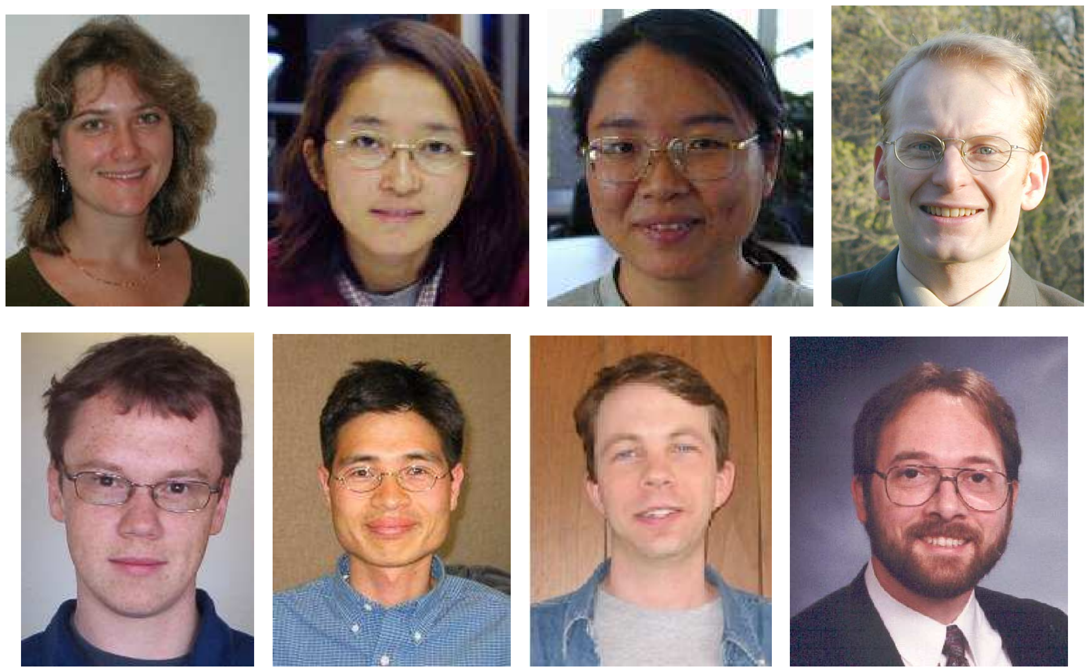


Figure 1: First level $\tilde{d}$ CAMP coefficients, organized by cosets.

Wisconsin



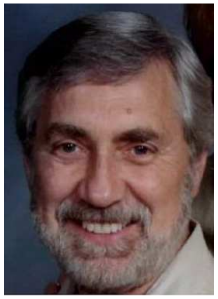
From left, 1st row:

Julia Velikina, Youngmi Hur, Yeon Kim, Narfi Stefansson.

2nd row:

Thomas Hangelbroek, Sangnam Nam, Jeff Kline, Steven Parker.

Julia Velikina: undersampled MRI data

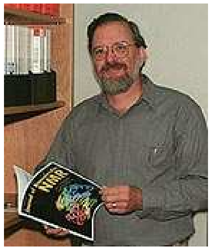
Schepp−Logan phantom

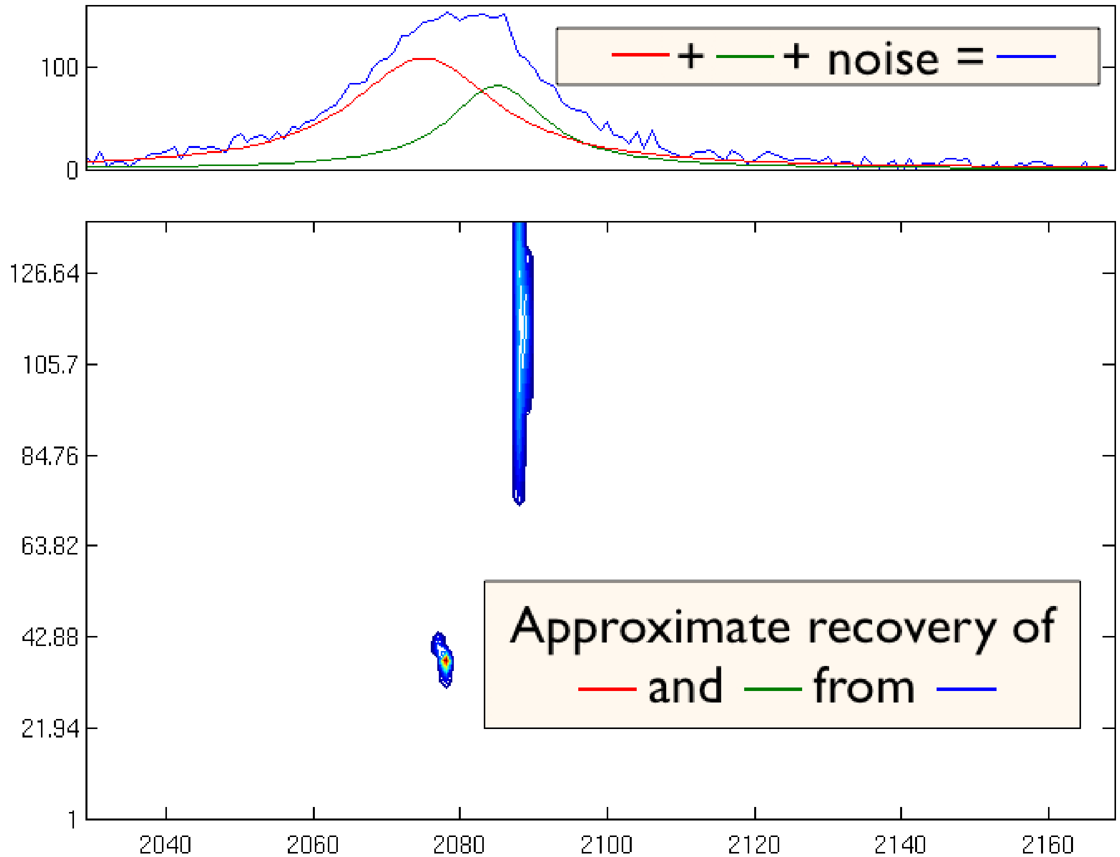
Conventional recon. from 90 projections, acceptable quality

Conventional recon. from 23 projections, unacceptable quality

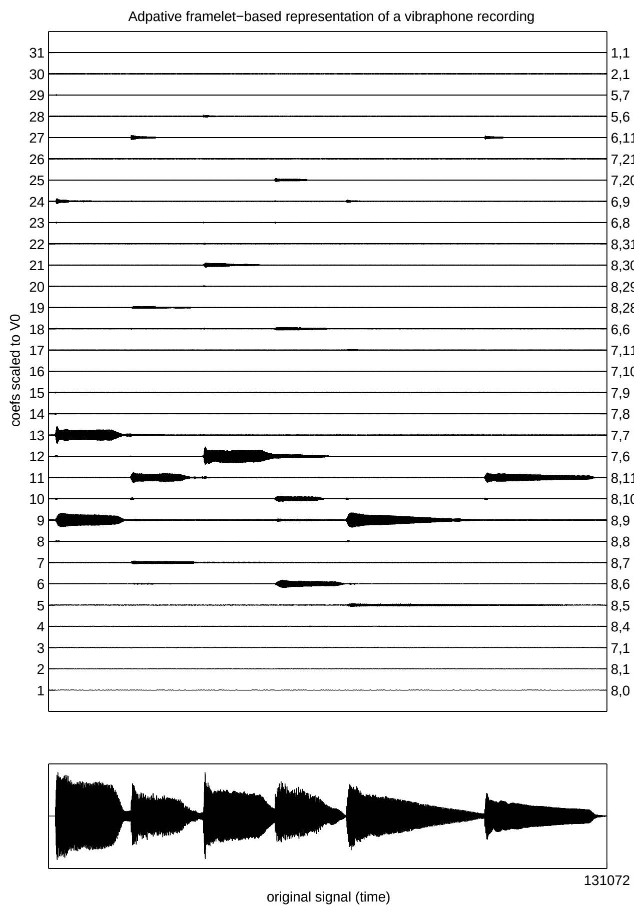
TV−based recon. from 23 projections

Jeff Kline: new data representation in NMR

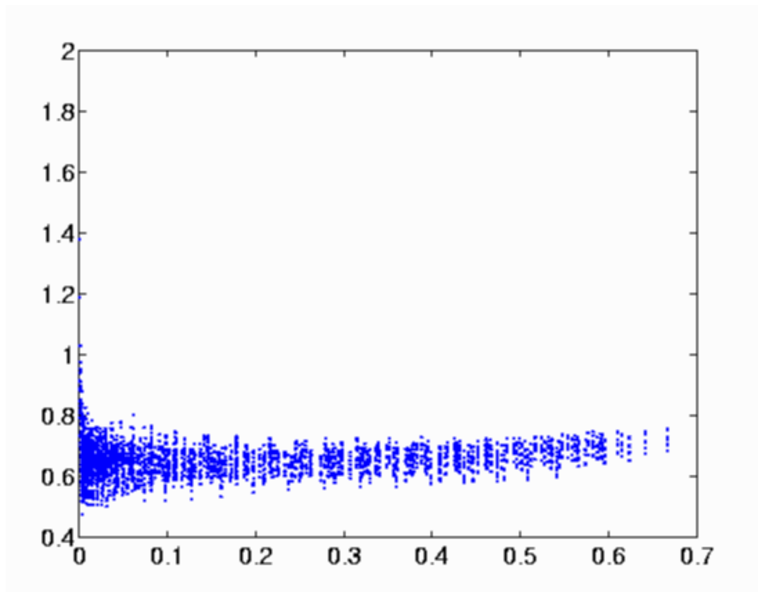





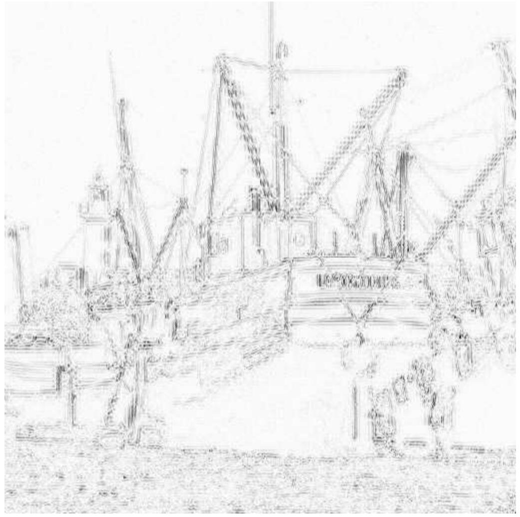
Steven Parker: redundant representation of acoustic signals



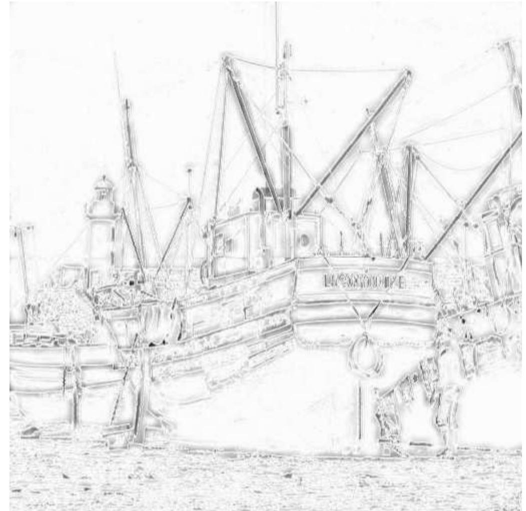
Narfi Stefansson: sparse framelet representations



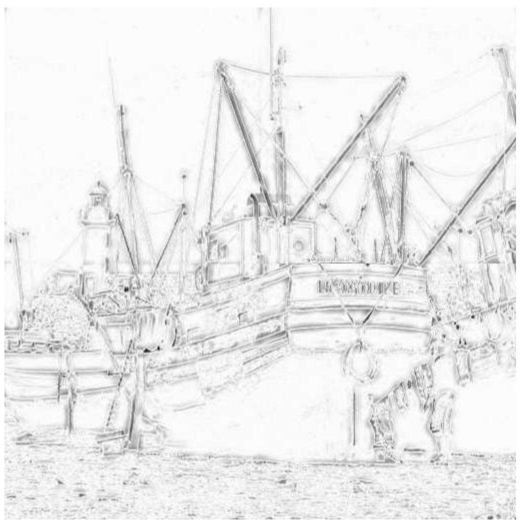
6/10 61440 coefficients



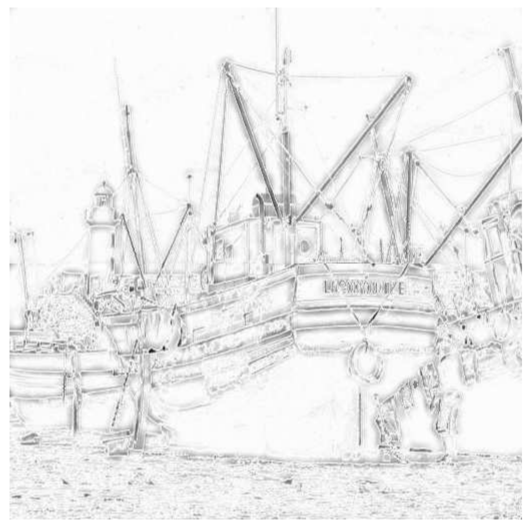
quartic spline 34452 coefficients



cubic spline 34608 coefficients



box15,box17,box18 35085 coefficients



FrameNet: on-line interactive framelet and wavelet analysis

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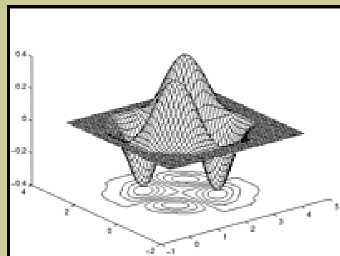


Version 1.0 Beta, October 2003

Welcome to **The IDR FrameNet Portal**, a web-based, research and educational tool for time/frequency analysis of data. If you are new to this site, we encourage you to take the tour or visit the Site Help.

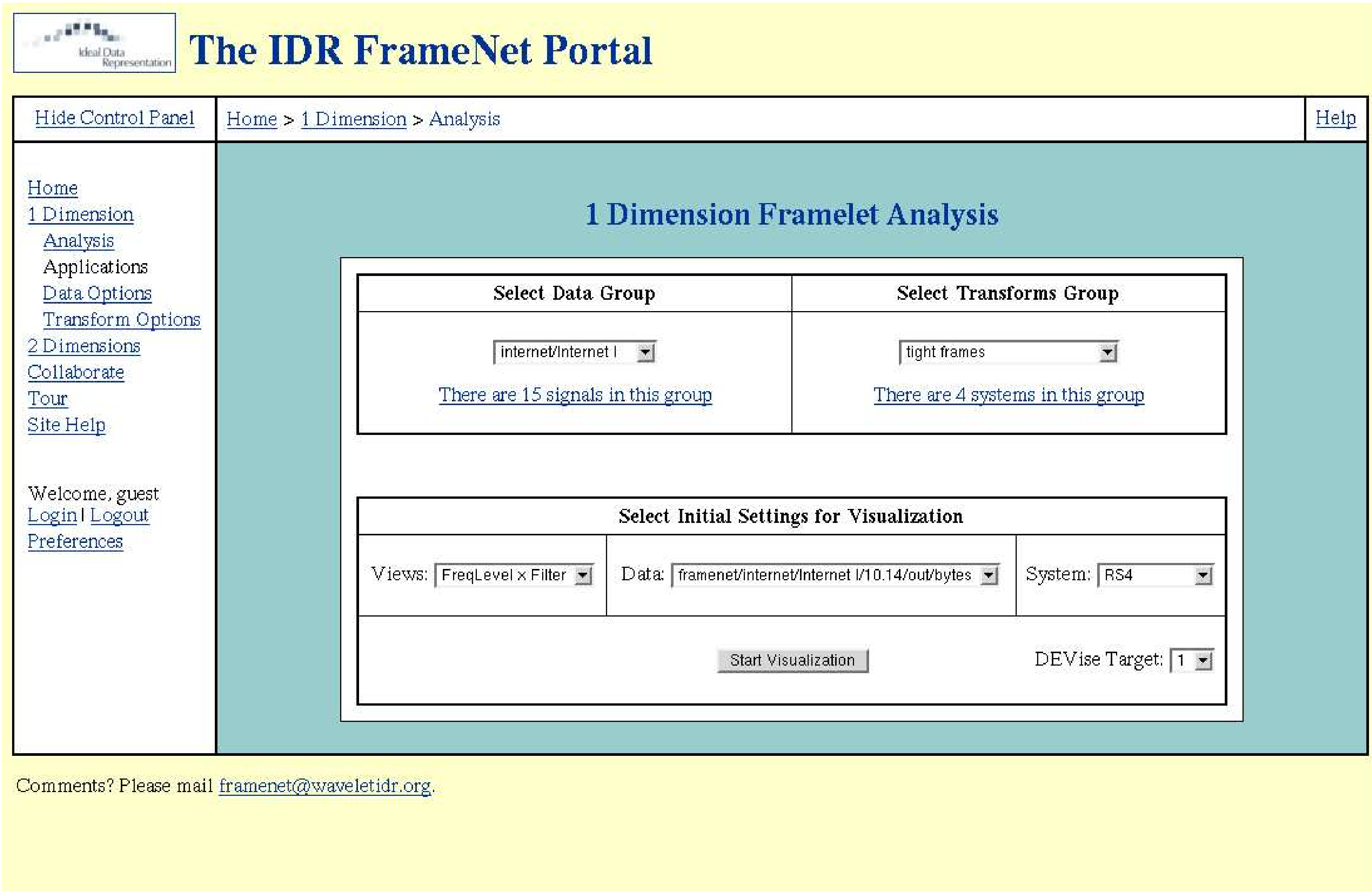
This tool provides facilities for uploading and management of scientific data, as well as dozens of available data sets from a variety of sources. Time/frequency analysis can be performed by classic wavelet systems, as well as by framelet systems (giving a redundant time/frequency description.) Furthermore, the FrameNet provides a collaboration mode, allowing researchers to work together on projects, and educators to demonstrate framelet analysis to their classes.

Group Leader: Amos Ron
Development Team Leader: Steven Parker
Development Team: Thomas Hangelbroek, Youngmi Hur, Jeff Kline, Narfi Stefansson, Bee-Chung Chen
with contributions from Carl de Boor, Miron Livny, Kent Wenger and Remi Gribonval.

This site is a project of the Wavelet Center for Ideal Data Representation. It incorporates the DEVise data exploration system and the LastWave signal processing software. Web hosting is maintained by Computer Systems Lab of the Computer Sciences Department, University of Wisconsin - Madison.

To contact the FrameNet team, send mail to framenet@waveletidr.org.

Comments? Please mail framenet@waveletidr.org.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Views: FreqLevel x Filter
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System: RS4
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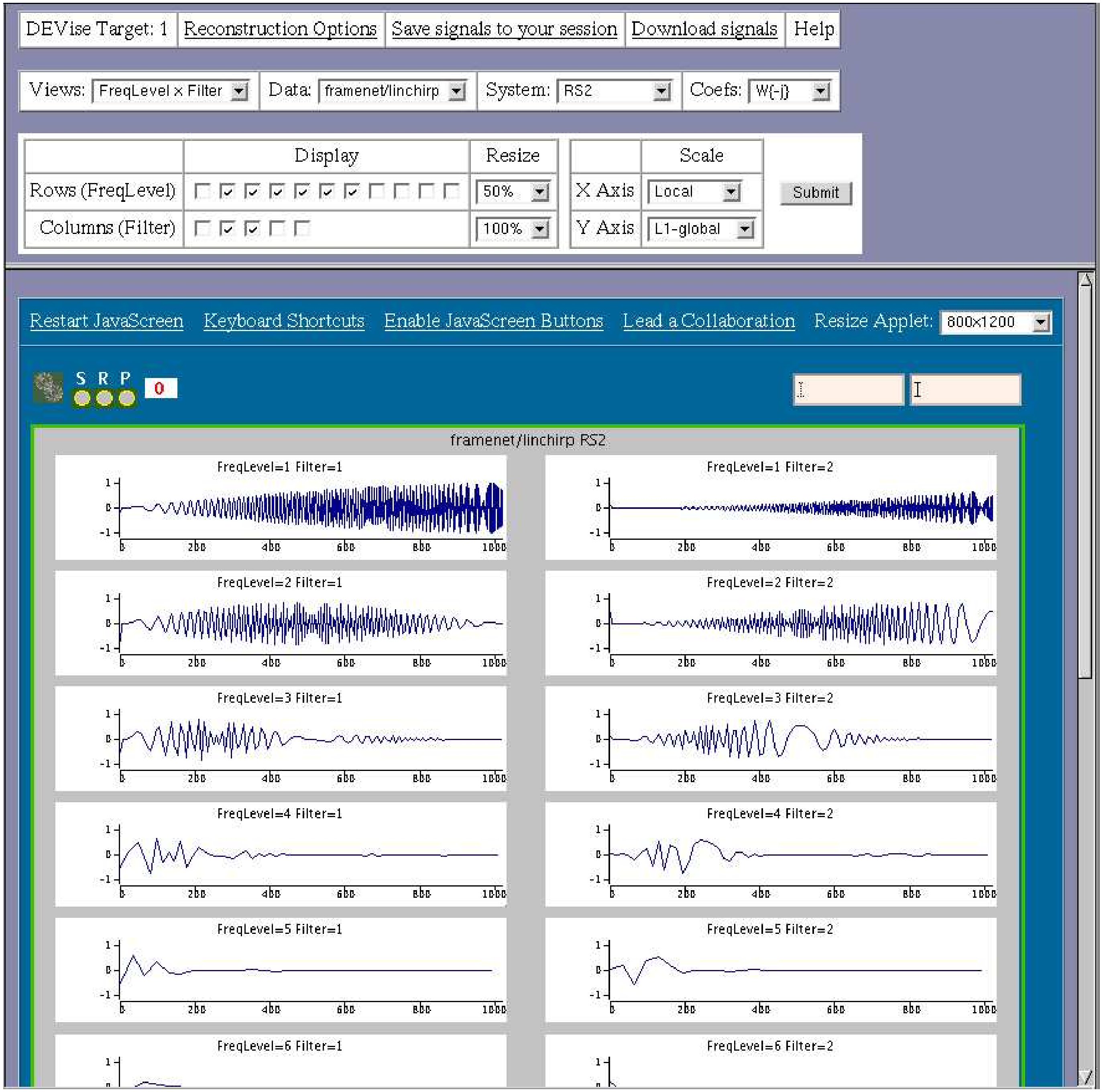
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Help
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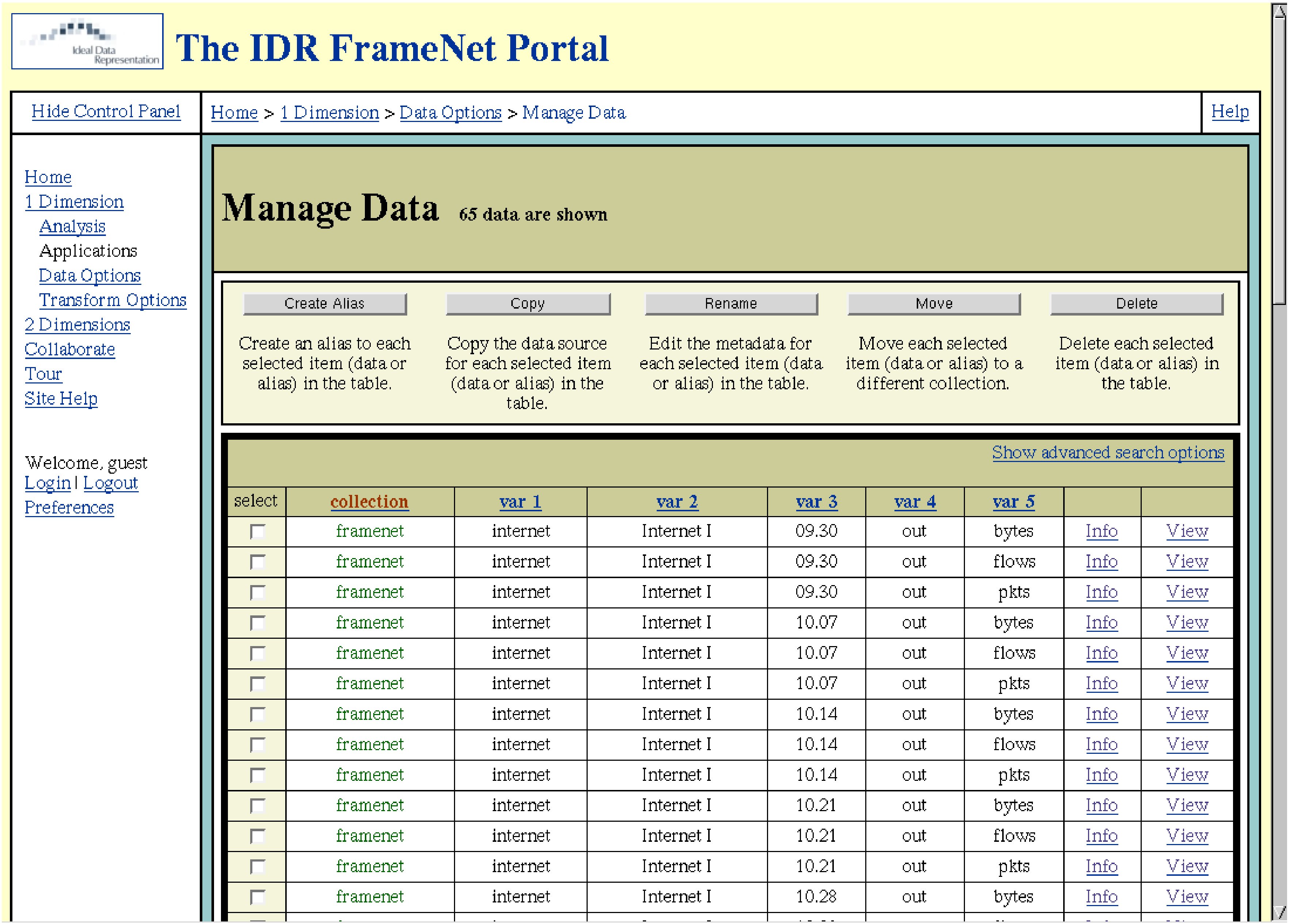
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