How the *k*-core decomposition helps in understanding the Internet Topology

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Outline

1. *k*-core decomposition
   - Definition
   - Examples
   - Applicability of *k*-cores
2. Applications
   - Properties analysis
   - Temporal analysis
   - Visualization

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k-core decomposition
Definition

Given $G = \{V, E\}$ a **undirected** graph, where V is the vertices set and E is the edges set.

Definition (Seidman, 1983 [4]) :

A subgraph $H = (C, E|C)$ induced by the set $C \subseteq V$ is a k-core or a core of order k iff $\forall v \in C : \mathtt{degree}_H(v) \geq k$, and H is the maximum subgraph with this property.

Then, a **minimal degree** k is imposed to the core of order k.

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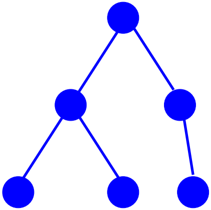
k-core decomposition
Examples

tree : 1-core



k-core decomposition
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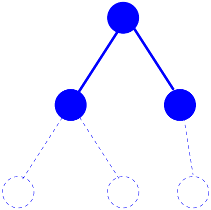
tree : 2-core ?



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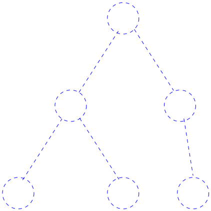

k-core decomposition
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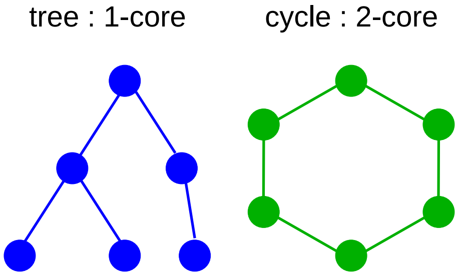

k-core decomposition
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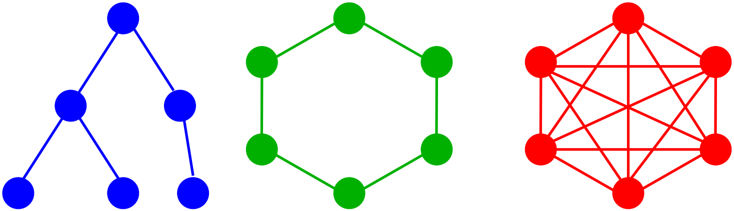
tree : 1-core — cycle : 2-core — clique n : $(n-1)$-core



k-core decomposition
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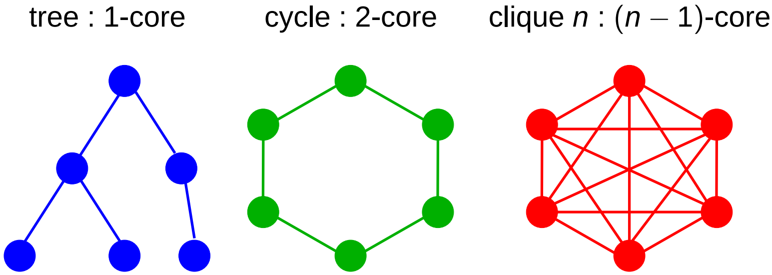
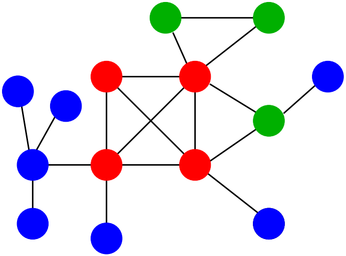


Remark : a *k*-connected graph **is** a *k*-core.

k-core decomposition
Examples

A graph :



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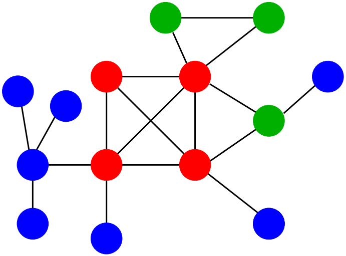

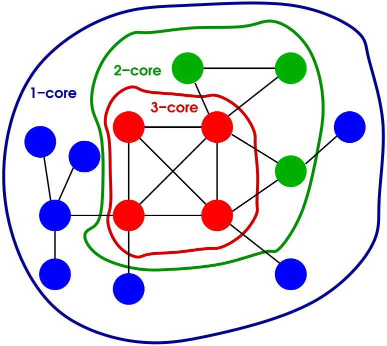

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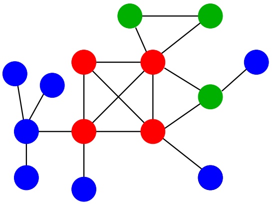

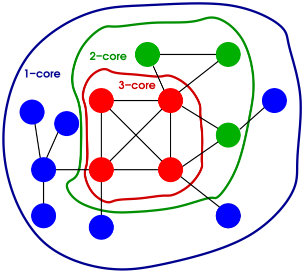

Definition

A vertex i has a *shell index c*, if it belongs to the c-core but not to $(c+1)$-core.

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Why the connectivity is interesting ?

k-core decomposition

The **connectivity** is mainly related to :

- robustness
  - faults,
  - attacks
- routing (to find a path between two vertices)
  - QoS,
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- Some results on *k*-core and *k*-connectivity :
  - Bollobas [1] : a **random graph** with large $< d >$ has a **minimum degree** *k* when $n \to \infty$, and it is *k*-connected.
  - Dimes project : study [2] the number of distinct paths as function of *k*-shell.
  - ***k*-edge-connectivity** : we show a theorem (k_{max}-core has diameter 2, and sets of vertices in *s*-shell are connected with at least *s* edges to higher cores).

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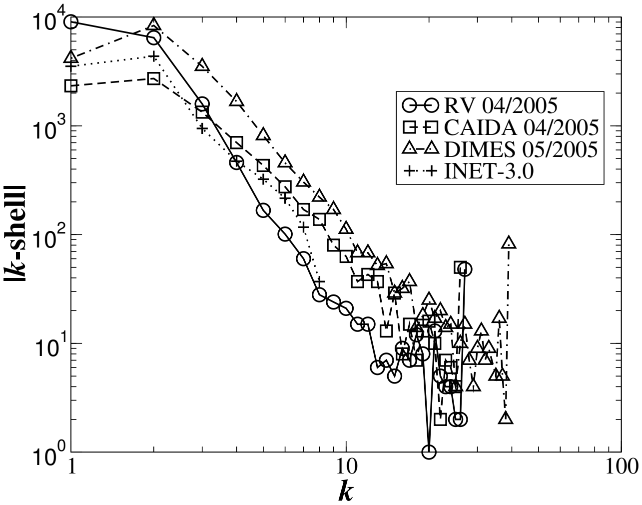
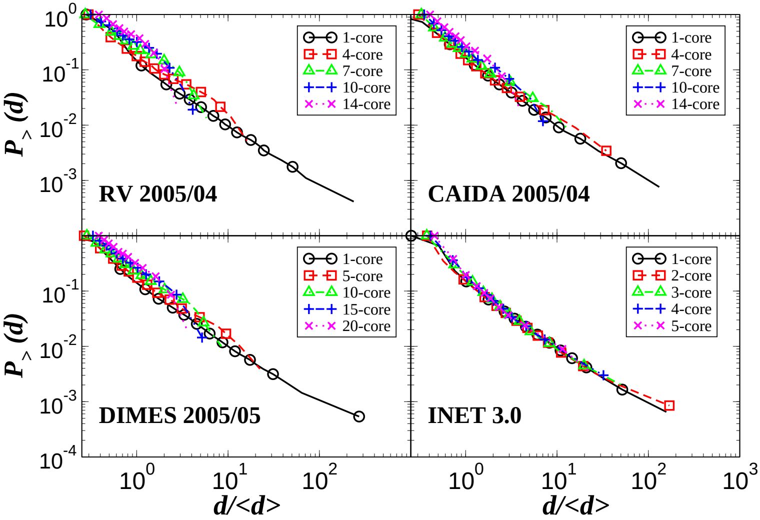
k-shell size vs *k*



cumulative degree distribution



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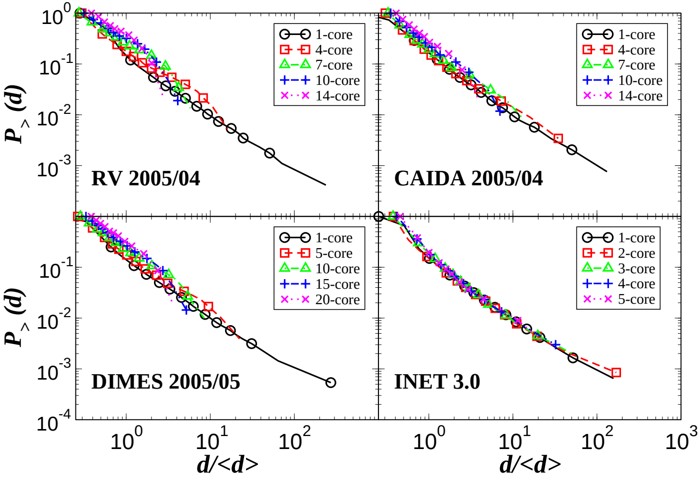
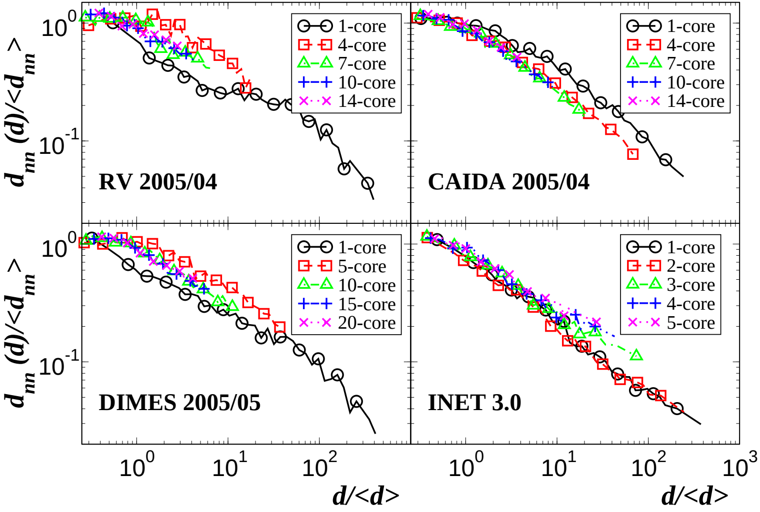
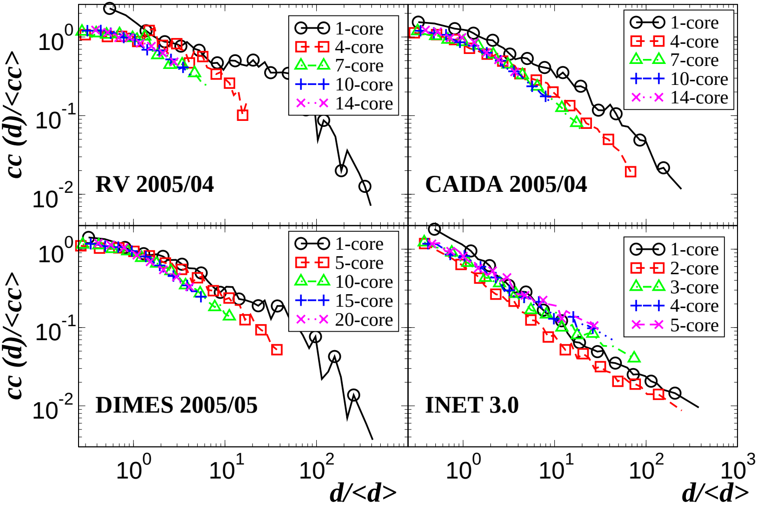


- **Self-similarity** of the *k*-**cores**

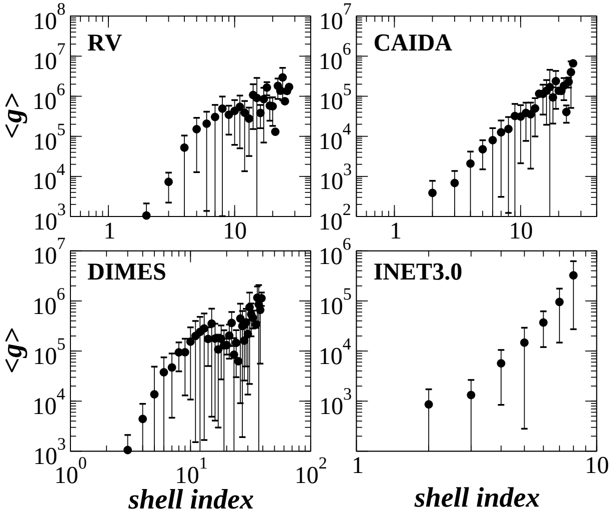
average neighbor degree vs degree



cluster coefficient vs degree



average betweenness vs shell index



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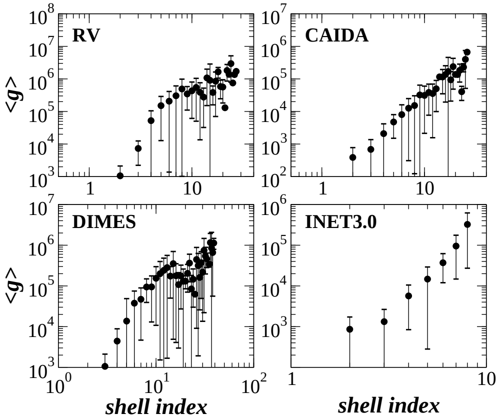

betweenness : $\mathcal{O}(n^2 \log n)$
k-core : $\mathcal{O}(e)$

k-**core** decomposition *establish* a **hierarchy** in the network :

- it can *identify* the vertex's **centrality**.

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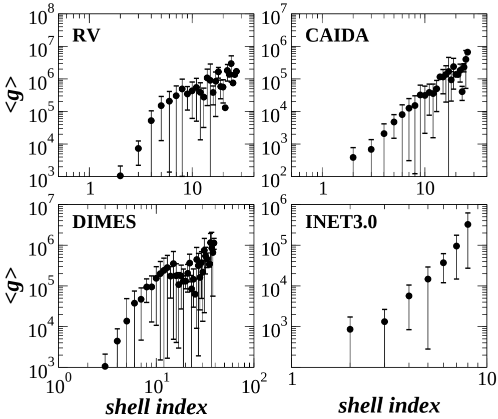

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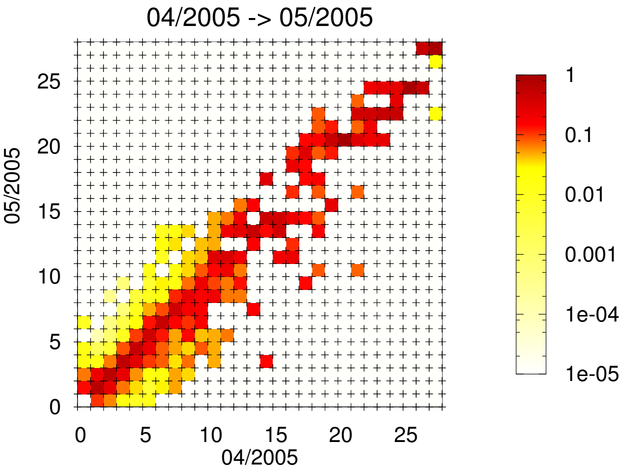
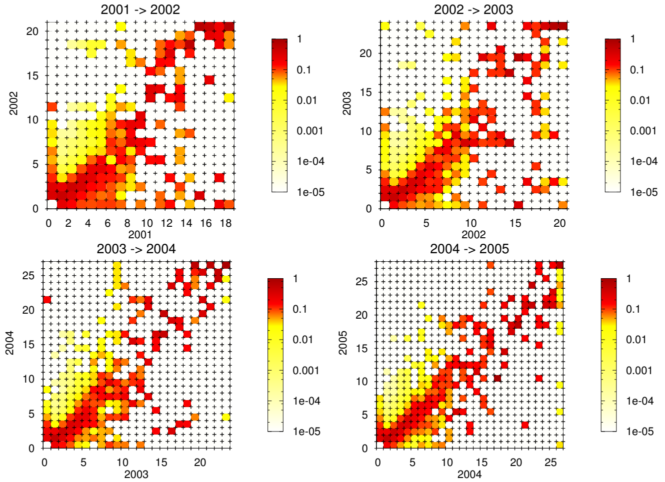
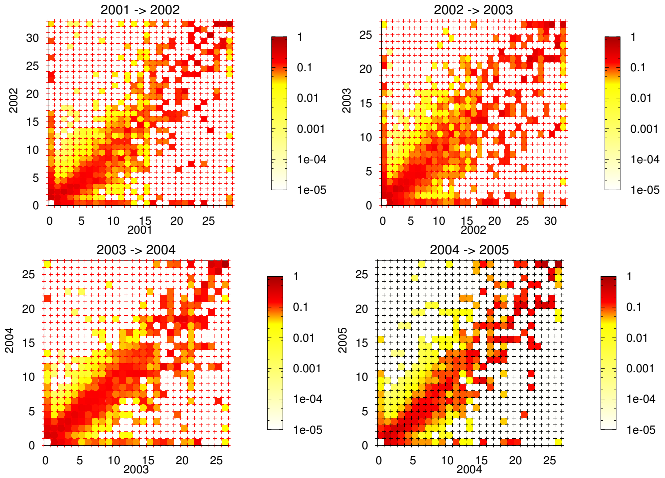
AS Oregon R. V. : april -> may 2005



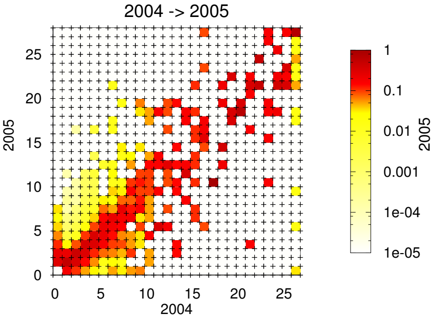
AS maps of Oregon R. V.



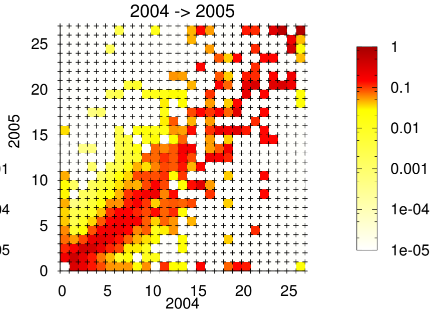
AS maps of CAIDA



AS maps : 2004->2005



Oregon_RV



CAIDA

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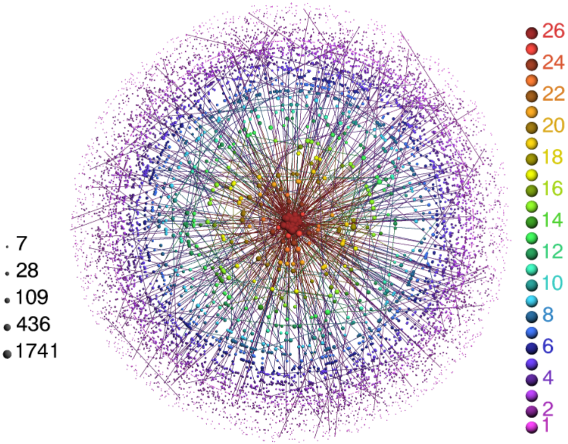
Network **fingerprints**



CAIDA AS map

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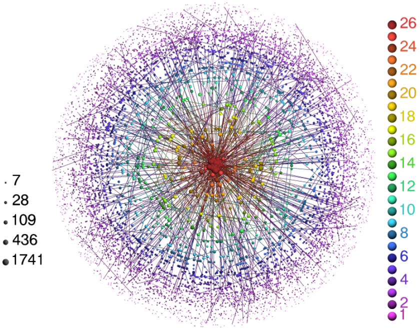

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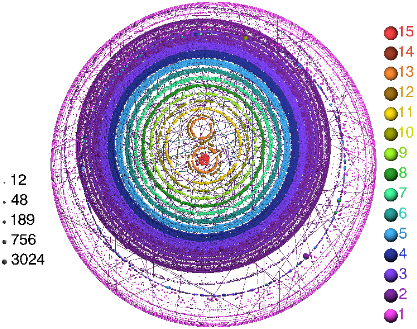

a portion of www.***.fr

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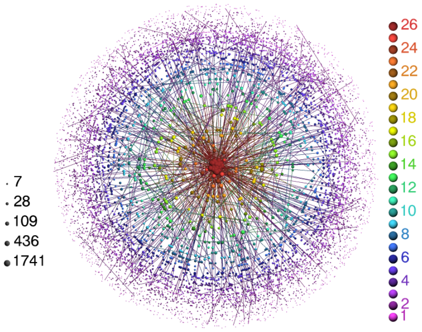

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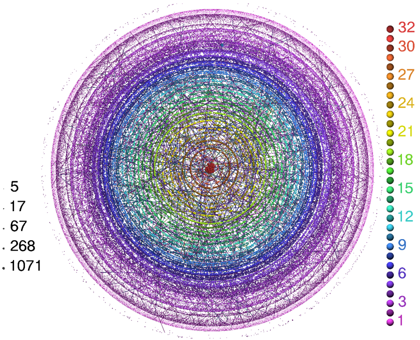

CAIDA Router level map

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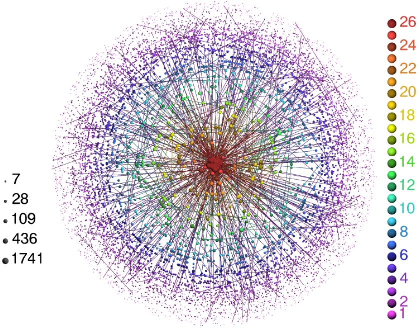

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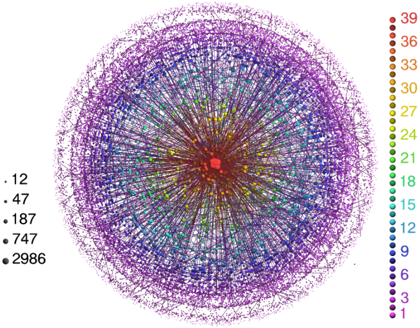

Dimes AS map

Summary

- Using *k*-core decomposition, we can determine :
  - **a <u>hierarchical</u> and <u>self-similar</u> structure**
  - **centrality**
  - **connectivity**
- **Comparative** analysis of **different** maps
- **Fingerprints** (Visualization)
  http ://xavier.informatics.indiana.edu/lanet-vi/
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k-edge-connectivity

- k_{max}-core is *k*-edge-connected (or diameter 2)
- cluster in *s*-shell is *s*-edge-connected and it has at least *s* edges towards $(s+1)$-core.
- then, the graph is *k*-edge-connected